Nome do aluno | Nº | Data / / 20

Fórmulas trigonométricas

1. Sabe-se que:

- $\theta \in \left]0, \frac{\pi}{4}\right[$
- $\sin\theta + \cos\theta = \frac{7}{5}$

Determine o valor exato de $\cos\left(\frac{\pi}{4} - \theta\right)$.

2. Determine o valor exato de $\sin\frac{\pi}{12}$.

3. Determine o valor exato de:

3.1. $\sin\frac{\pi}{5}\cos\frac{2\pi}{15} + \cos\frac{\pi}{5}\sin\frac{2\pi}{15}$

3.2. $\cos\frac{7\pi}{12} - \cos\frac{\pi}{12}$

3.3. $\frac{\tan 117^\circ + \tan 18^\circ}{1 - \tan 117^\circ \tan 18^\circ}$

4. Mostre que para qualquer x real:

$$\sin\left(\frac{\pi}{4} + x\right) + \cos\left(\frac{\pi}{4} + x\right) = \sqrt{2}\cos x$$

5. Sabendo que:

$$\sin x = \frac{12}{13}\ \left(x \in \left]\frac{\pi}{2}, \pi\right[\right) \qquad e \qquad \cos y = \frac{1}{2}\ \left(y \in \left]-\frac{\pi}{2}, 0\right[\right)$$

Determine o valor exato de:

$$\cos(x + y)$$

6. Resolva, em $\mathbb{R}$, as equações seguintes:

6.1. $\sin x \cos\frac{\pi}{7} + \sin\frac{\pi}{7}\cos x = \frac{1}{2}$

6.2. $\frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x = 1$

6.3. $3\cos x + \sqrt{3}\sin x = -3$

6.4. $\sin x + \cos x = -\frac{\sqrt{2}}{2}$

7. Prove que

$$\tan\left(x + \frac{\pi}{4}\right) = \frac{1 + \tan x}{1 - \tan x}$$

E indique o maior subconjunto dos números reais onde a igualdade é possível.

8. Resolva, em $]-\pi, \pi[$:

8.1. $\tan\left(x + \frac{\pi}{4}\right) = \sqrt{3}$

8.2. $\tan\left(x + \frac{\pi}{4}\right) = 1 + \tan x$

9. Considere um retângulo em que se sabe que:

- A sua diagonal mede 2 cm;
- x é a amplitude do ângulo entre o segmento da diagonal e um dos lados maiores do retângulo.

9.1. Mostre que a área do retângulo é dada por:

$$A(x) = 2\sin(2x)$$

9.2. Mostre que o perímetro do retângulo é dado por:

$$P(x) = 4\sqrt{2}\cos\left(x - \frac{\pi}{4}\right)$$

10. Mostre que:

10.1. $\cos(2x) = 1 - 2\sin^2 x$

10.2. $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$

10.3. $\tan x + \frac{1}{\tan x} = \frac{2}{\sin(2x)}$, $\forall\, x \neq k\frac{\pi}{2}, k \in \mathbb{Z}$

11. Determine os zeros da função f, de domínio $]-\pi, \pi[$, definida por:

$$f(x) = 1 + 4\sin x\cos x$$

12. Resolva, em $\mathbb{R}$, a equação:

$$1 - \cos x = \sin\frac{x}{2}$$

Repare que: $x = 2\frac{x}{2}$

13. Considere o triângulo $[ABC]$ representado na figura seguinte.
Sabe-se que:

- $[ABC]$ é um triângulo isósceles em que $\overline{AC} = \overline{BC} = 1$;
- x designa a amplitude do ângulo BAC.

Mostre que a área do triângulo $[ABC]$ é dada, em função de x por:

$$A(x) = \frac{1}{2}\sin(2x)$$

Soluções

1.

$$\cos\left(\frac{\pi}{4}-\theta\right)=\cos\frac{\pi}{4}\cos\theta+\sin\frac{\pi}{4}\sin\theta \Leftrightarrow$$

$$\Leftrightarrow \cos\left(\frac{\pi}{4}-\theta\right)=\frac{\sqrt{2}}{2}\cos\theta+\frac{\sqrt{2}}{2}\sin\theta \Leftrightarrow$$

$$\Leftrightarrow \cos\left(\frac{\pi}{4}-\theta\right)=\frac{\sqrt{2}}{2}(\cos\theta+\sin\theta) \Leftrightarrow$$

$$\Leftrightarrow \cos\left(\frac{\pi}{4}-\theta\right)=\frac{\sqrt{2}}{2}\times\frac{7}{5} \Leftrightarrow$$

$$\Leftrightarrow \cos\left(\frac{\pi}{4}-\theta\right)=\frac{7\sqrt{2}}{10}$$

2.

$$\sin\frac{\pi}{12}=\sin\left(\frac{\pi}{4}-\frac{\pi}{6}\right)=\sin\frac{\pi}{4}\cos\frac{\pi}{6}-\cos\frac{\pi}{6}\sin\frac{\pi}{4}=$$

$$=\frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\times\frac{1}{2}=\frac{\sqrt{6}-\sqrt{2}}{4}$$

3.

3.1.

$$\sin\frac{\pi}{5}\cos\frac{2\pi}{15}+\cos\frac{\pi}{5}\sin\frac{2\pi}{15}=\sin\left(\frac{\pi}{5}+\frac{2\pi}{15}\right)=\sin\left(\frac{3\pi}{15}+\frac{2\pi}{15}\right)=\sin\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$$

3.2.

$$\cos\frac{7\pi}{12}-\cos\frac{\pi}{12}=\cos\left(\frac{\pi}{4}+\frac{\pi}{3}\right)-\cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right)=$$

$$=\cos\frac{\pi}{4}\cos\frac{\pi}{3}-\sin\frac{\pi}{4}\sin\frac{\pi}{3}-\cos\frac{\pi}{4}\cos\frac{\pi}{6}-\sin\frac{\pi}{4}\sin\frac{\pi}{6}=$$

$$=\frac{\sqrt{2}}{2}\times\frac{1}{2}-\frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\times\frac{1}{2}=-\frac{\sqrt{6}}{2}$$

3.3.

$$\frac{\tan 117^\circ+\tan 18^\circ}{1-\tan 117^\circ\tan 18^\circ}=\tan(117^\circ+18^\circ)=\tan 135^\circ=-1$$

4.

Seja $x\in\mathbb{R}$

$$\sin\left(\frac{\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}+x\right)=$$

$$=\sin\frac{\pi}{4}\cos x+\cos\frac{\pi}{4}\sin x+\cos\frac{\pi}{4}\cos x-\sin\frac{\pi}{4}\sin x=$$

$$=\frac{\sqrt{2}}{2}\cos x+\frac{\sqrt{2}}{2}\sin x+\frac{\sqrt{2}}{2}\cos x-\frac{\sqrt{2}}{2}\sin x=\sqrt{2}\cos x$$

5.

Pela fórmula fundamental da trigonometria:

$$\sin^2 x + \cos^2 x = 1 \Leftrightarrow \left(\frac{12}{13}\right)^2 + \cos^2 x = 1 \Leftrightarrow \cos^2 x = 1 - \frac{144}{169} \Leftrightarrow \cos^2 x = \frac{25}{169} \Leftrightarrow \cos^2 x = \pm\frac{5}{13}$$

como $x \in \left]\frac{\pi}{2}, \pi\right[$, $\cos x < 0$, então, $\cos x = -\frac{5}{13}$

$$\sin^2 y + \cos^2 y = 1 \Leftrightarrow \sin^2 y + \left(\frac{1}{2}\right)^2 = 1 \Leftrightarrow \sin^2 y = 1 - \frac{1}{4} \Leftrightarrow \sin^2 y = \frac{3}{4} \Leftrightarrow \sin y = \pm\frac{\sqrt{3}}{2}$$

como $y \in \left]-\frac{\pi}{2}, 0\right[$, $\sin y < 0$, então, $\sin y = -\frac{\sqrt{3}}{2}$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y = -\frac{5}{13} \times \frac{1}{2} + \frac{12}{13} \times \frac{\sqrt{3}}{2} = \frac{12\sqrt{3} - 5}{26}$$

6.

6.1.

$$\sin x \cos\frac{\pi}{7} + \sin\frac{\pi}{7}\cos x = \frac{1}{2} \Leftrightarrow \sin\left(x + \frac{\pi}{7}\right) = \frac{1}{2} \Leftrightarrow$$

$$\Leftrightarrow \sin\left(x + \frac{\pi}{7}\right) = \sin\left(\frac{\pi}{6}\right) \Leftrightarrow$$

$$\Leftrightarrow x + \frac{\pi}{7} = \frac{\pi}{6} + 2k\pi \vee x + \frac{\pi}{7} = \pi - \frac{\pi}{6} + 2k\pi, k \in \mathbb{Z} \Leftrightarrow$$

$$\Leftrightarrow x = \frac{\pi}{42} + 2k\pi \vee x = \frac{29\pi}{42} + 2k\pi, k \in \mathbb{Z}$$

$$\text{C.S.} = \left\{x\colon x = \frac{\pi}{42} + 2k\pi \vee x = \frac{29\pi}{42} + 2k\pi, k \in \mathbb{Z}\right\}$$

6.2.

$$\frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x = 1 \Leftrightarrow \cos\frac{\pi}{3}\sin x - \sin\frac{\pi}{3}\cos x = 1 \Leftrightarrow$$

$$\Leftrightarrow \sin\left(x - \frac{\pi}{3}\right) = \sin\frac{\pi}{2} \Leftrightarrow x - \frac{\pi}{3} = \frac{\pi}{2} + 2k\pi, k \in \mathbb{Z} \Leftrightarrow$$

$$\Leftrightarrow x = \frac{5\pi}{6} + 2k\pi, k \in \mathbb{Z}$$

$$\text{C.S.} = \left\{x\colon x = \frac{5\pi}{6} + 2k\pi, k \in \mathbb{Z}\right\}$$

6.3.

$$3\cos x + \sqrt{3}\sin x = -3 \Leftrightarrow \cos x + \frac{\sqrt{3}}{3}\sin x = -1 \Leftrightarrow$$

$$\Leftrightarrow \cos x + \tan\left(\frac{\pi}{6}\right)\sin x = -1 \Leftrightarrow \cos x + \frac{\sin\left(\frac{\pi}{6}\right)}{\cos\left(\frac{\pi}{6}\right)}\sin x = -1 \Leftrightarrow$$

$$\Leftrightarrow \cos\left(\frac{\pi}{6}\right)\cos x + \sin\left(\frac{\pi}{6}\right)\sin x = -\cos\left(\frac{\pi}{6}\right) \Leftrightarrow$$

$$\Leftrightarrow \cos\left(\frac{\pi}{6} - x\right) = \cos\left(\pi + \frac{\pi}{6}\right) \Leftrightarrow \cos\left(\frac{\pi}{6} - x\right) = \cos\left(\frac{7\pi}{6}\right) \Leftrightarrow$$

$$\Leftrightarrow \frac{\pi}{6} - x = \frac{7\pi}{6} + 2k\pi \vee \frac{\pi}{6} - x = -\frac{7\pi}{6} + 2k\pi, k \in \mathbb{Z}$$

$$x = -\pi + 2k\pi \vee x = \frac{4\pi}{3} + 2k\pi, k \in \mathbb{Z}$$

$$\text{C.S.} = \left\{x\text{: } x = -\pi + 2k\pi \vee x = \frac{4\pi}{3} + 2k\pi, k \in \mathbb{Z}\right\}$$

6.4.

$$\sin x + \cos x = -\frac{\sqrt{2}}{2} \Leftrightarrow \frac{\sqrt{2}}{2}\sin x + \frac{\sqrt{2}}{2}\cos x = -\left(\frac{\sqrt{2}}{2}\right)^2 \Leftrightarrow$$

$$\Leftrightarrow \sin\frac{\pi}{4}\sin x + \cos\frac{\pi}{4}\cos x = -\frac{1}{2} \Leftrightarrow \cos\left(x - \frac{\pi}{4}\right) = \cos\left(\pi - \frac{\pi}{3}\right) \Leftrightarrow$$

$$\Leftrightarrow x - \frac{\pi}{4} = \frac{2\pi}{3} + 2k\pi \vee x - \frac{\pi}{4} = -\frac{2\pi}{3} + 2k\pi, k \in \mathbb{Z} \Leftrightarrow$$

$$\Leftrightarrow x = \frac{2\pi}{3} + \frac{\pi}{4} + 2k\pi \vee x = -\frac{2\pi}{3} + \frac{\pi}{4} + 2k\pi, k \in \mathbb{Z} \Leftrightarrow$$

$$\Leftrightarrow x = \frac{11\pi}{12} + 2k\pi \vee x = -\frac{5\pi}{12} + 2k\pi, k \in \mathbb{Z}$$

$$\text{C.S.} = \left\{x\text{: } x = \frac{11\pi}{12} + 2k\pi \vee x = -\frac{5\pi}{12} + 2k\pi, k \in \mathbb{Z}\right\}$$

7.

$$\tan\left(x + \frac{\pi}{4}\right) = \frac{\tan x + \tan\frac{\pi}{4}}{1 - \tan x \tan\frac{\pi}{4}} = \frac{\tan x + 1}{1 - \tan x \times 1} = \frac{1 + \tan x}{1 - \tan x}$$

$$1 - \tan x = 0 \Leftrightarrow \tan x = 1 \Leftrightarrow \tan x = \tan\frac{\pi}{4} \Leftrightarrow x = \frac{\pi}{4} + k\pi, k \in \mathbb{Z}$$

$$x \neq \frac{\pi}{4} + k\pi, k \in \mathbb{Z} \text{ e } x \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$$

$$\mathbb{R}\backslash\left\{x = \frac{\pi}{4} + k\pi \vee x = \frac{\pi}{2} + k\pi, k \in \mathbb{Z}\right\}$$

8.

8.1.

$$\tan\left(x + \frac{\pi}{4}\right) = \sqrt{3} \Leftrightarrow \tan\left(x + \frac{\pi}{4}\right) = \tan\left(\frac{\pi}{3}\right) \Leftrightarrow$$

$$\Leftrightarrow x + \frac{\pi}{4} = \frac{\pi}{3} + k\pi, k \in \mathbb{Z} \Leftrightarrow x = \frac{\pi}{12} + k\pi, k \in \mathbb{Z}$$

$$k = 0 \rightarrow x = \frac{\pi}{12} \checkmark$$

$$k = 1 \rightarrow x = \frac{13\pi}{12} \times$$

$$k = -1 \rightarrow x = -\frac{11\pi}{12} \checkmark$$

$$\text{C.S.} = \left\{-\frac{11\pi}{12}, \frac{\pi}{12}\right\}$$

8.2.

$$\tan\left(x+\frac{\pi}{4}\right) = 1+\tan x \Leftrightarrow \frac{1+\tan x}{1-\tan x} = 1+\tan x \Leftrightarrow$$

$$\Leftrightarrow 1+\tan x = (1+\tan x)(1-\tan x) \Leftrightarrow$$

$$\Leftrightarrow \frac{1+\tan x}{1+\tan x} = 1-\tan x \Leftrightarrow$$

$$\Leftrightarrow 1+\tan x = 1 \Leftrightarrow \tan x = 1-1 \Leftrightarrow \tan x = 0$$

$$\text{C.S.} = \left\{-\frac{\pi}{4}, 0, \frac{3\pi}{4}\right\}$$

9.

9.1.

Sejam c e l o comprimento e a largura do retângulo, respetivamente.

$$\sin x = \frac{l}{2} \Leftrightarrow l = 2\sin x \qquad \cos x = \frac{c}{2} \Leftrightarrow c = 2\cos x$$

$$A(x) = 2\sin x \times 2\cos x = 2 \times (2\sin x \times \cos x) = 2\sin(2x)$$

9.2.

$$P(x) = 2 \times 2\sin x + 2 \times 2\cos x \Leftrightarrow P(x) = 4(\sin x + \cos x) \Leftrightarrow$$

$$\Leftrightarrow P(x) = 4\sqrt{2} \times \frac{\sqrt{2}}{2}(\sin x + \cos x) \Leftrightarrow$$

$$\Leftrightarrow P(x) = 4\sqrt{2}\left(\frac{\sqrt{2}}{2}\sin x + \frac{\sqrt{2}}{2}\cos x\right) \Leftrightarrow$$

$$\Leftrightarrow P(x) = 4\sqrt{2}\left(\cos x \cos\frac{\pi}{4} + \sin x \sin\frac{\pi}{4}\right) \Leftrightarrow P(x) = 4\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)$$

10.

10.1.

$$\cos(2x) = \cos^2 x - \sin^2 x = 1 - \sin^2 x - \sin^2 x = 1 - 2\sin^2 x$$

10.2.

$$\sin x = \sin\left(2 \times \frac{x}{2}\right) = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

10.3.

$$\tan x + \frac{1}{\tan x} = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{\sin^2 x + \cos^2 x}{\cos x \sin x} = \frac{1}{\sin x \cos x} =$$

$$= \frac{2}{2\cos x \sin x} = \frac{2}{\sin(2x)}$$

$$\forall x \neq \frac{k\pi}{2}, k \in \mathbb{Z}$$

11.

$$f(x) = 0 \Leftrightarrow 1 + 4\sin x \cos x = 0 \Leftrightarrow 1 + 2 \times 2\sin x \cos x = 0 \Leftrightarrow$$

$$\Leftrightarrow 1 + 2\sin(2x) = 0 \Leftrightarrow \sin(2x) = -\frac{1}{2} \Leftrightarrow$$

$$\Leftrightarrow 2x = -\frac{\pi}{6} + 2k\pi \vee 2x = \pi + \frac{\pi}{6} + 2k\pi, k \in \mathbb{Z}$$

$$x = -\frac{\pi}{12} + k\pi \vee x = \frac{7\pi}{12} + k\pi, k \in \mathbb{Z}$$

em $]-\pi, \pi[$:

$k = 0 \to x = -\frac{\pi}{12} \vee x = \frac{7\pi}{12}$

$k = 1 \to x = \frac{11\pi}{12} \vee x = \frac{19\pi}{12}$

$k = -1 \to x = -\frac{13\pi}{12} \vee x = -\frac{5\pi}{12}$

$f(x) = 0 \Leftrightarrow x \in \left\{-\frac{5\pi}{12}; -\frac{\pi}{12}; \frac{7\pi}{12}; \frac{11\pi}{12}\right\}$

12.

$$1 - \cos x = \sin\left(\frac{x}{2}\right) \Leftrightarrow 1 - \cos\left(2 \times \frac{x}{2}\right) = \sin\left(\frac{x}{2}\right) \Leftrightarrow$$

$$\Leftrightarrow 1 - \cos^2\left(\frac{x}{2}\right) + \sin^2\left(\frac{x}{2}\right) = \sin\left(\frac{x}{2}\right) \Leftrightarrow \sin^2\left(\frac{x}{2}\right) + \sin^2\left(\frac{x}{2}\right) = \sin\left(\frac{x}{2}\right) \Leftrightarrow$$

$$\Leftrightarrow 2\sin^2\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right) = 0 \Leftrightarrow \sin\left(\frac{x}{2}\right)\left[2\sin\left(\frac{x}{2}\right) - 1\right] = 0 \Leftrightarrow$$

$$\Leftrightarrow \sin\left(\frac{x}{2}\right) = 0 \vee \sin\left(\frac{x}{2}\right) = \frac{1}{2} \Leftrightarrow \sin\left(\frac{x}{2}\right) = \sin 0 \vee \sin\left(\frac{x}{2}\right) = \sin\left(\frac{\pi}{6}\right) \Leftrightarrow$$

$$\Leftrightarrow \frac{x}{2} = 2k\pi \vee \frac{x}{2} = \pi + 2k\pi \vee \frac{x}{2} = \frac{\pi}{6} + 2k\pi \vee \frac{x}{2} = \frac{5\pi}{6} + 2k\pi, k \in \mathbb{Z}$$

$$x = 4k\pi \vee x = 2\pi + 4k\pi \vee x = \frac{\pi}{3} + 4k\pi \vee x = \frac{5\pi}{3} + 4k\pi, k \in \mathbb{Z}$$

13.

Seja h a altura do triângulo relativa à base $[AB]$.

Então, $h = \sin x$ e $\overline{AB} = 2\cos x$.

Assim: $A(x) = \frac{\sin(2x)}{2}$