Nome do aluno | Nº | Data / / 20

Operações com números complexos na forma trigonométrica

1. Considere os números complexos:

$$z_1 = 2e^{-i\frac{2\pi}{5}} \qquad z_2 = -5\sqrt{3} + 5i \qquad z_3 = 2i$$

Calcule, apresentando o resultado na forma trigonométrica:

1.1. $z_1 z_3$

1.2. $z_2 \overline{z_1}$

1.3. $\frac{z_1}{z_3}$

1.4. $\frac{iz_1}{-z_2}$

1.5. $z_1^2 \times i^{22}$

2. Sejam A e B os afixos dos números complexos unitários $z_0 = 1$ e $z = a + bi, a, b \in \mathbb{R}$.
Determine os valores de a e b, de modo que:

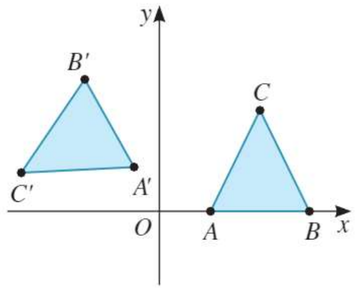
2.1. $A\hat{O}B = \frac{2\pi}{3}$

2.2. $A\hat{O}B = \frac{3\pi}{2}$

2.3. $A\hat{O}B = \frac{5\pi}{4}$

3. No referencial ortonormado direto da figura está representado o triângulo cujos vértices A, B e C são os afixos dos números complexos 1, 3 e $2\sqrt{2}e^{i\frac{\pi}{4}}$, respetivamente, e o seu transformado $[A'B'C']$ por uma rotação de centro na origem O e amplitude $\frac{2\pi}{3}$.

Determine, na forma trigonométrica, os números complexos cujos afixos são os vértices de $[A'B'C']$.



4. Considere o número complexo:

$$z = e^{i\theta}, \theta \in \mathbb{R}$$

4.1. Determine, em função de θ, um argumento de:

4.1.1. iz

4.1.2. $\frac{\bar{z}}{3i}$

4.1.3. $(\sqrt{2} - \sqrt{2}i)z$

4.2. Admita que $\theta = \frac{5\pi}{6}$.
Identifique a transformação geométrica que transforma o afixo de z no afixo do complexo $w = -\sqrt{3} - i$.

5. Calcule, apresentando o resultado na forma $a + bi$ com $a, b \in \mathbb{R}$:

5.1. $\left(2e^{i\frac{\pi}{4}}\right)^5$

5.2. $\left(\sqrt{3} - i\right)^{10}$

5.3. $\left(\frac{2e^{i\frac{\pi}{6}}}{3i}\right)^{-7}$

6. Calcule:

$$\left|\frac{(1+i)^5}{i(1-i)^3}\right|$$

7. Seja z um número complexo cujo afixo pertence ao primeiro quadrante (eixos não incluídos).

Justifique que a imagem geométrica de z^3 não pode pertencer ao quarto quadrante.

8. Em $\mathbb{C}$, conjunto dos números complexos, considere os números:

$$z_1 = -1 + i \qquad\qquad z_2 = 2e^{i\frac{\pi}{6}}$$

8.1. Mostre que $\frac{\overline{z_1^4-2}}{3i}$ é um imaginário puro.

8.2. Determine, na forma trigonométrica, z, não nulo, tal que:

$$z^3 z_1 = \frac{z}{\overline{z_2}}$$

9. Determine os valores de $r > 0$ e $\theta \in]-\pi, \pi]$, tais que:

9.1. $\left(re^{i\theta}\right)^2 = -2$

9.2. $\left(re^{i\theta}\right)^3 = -2$

9.3. $\left(re^{i\theta}\right)^4 = -2$

10. Resolva em $\mathbb{C}$ as seguintes equações:

10.1. $z^4 - 1{=}0$

10.2. $z^3 = 27i$

10.3. $z^2 + 16i = 0$

10.4. $z^3 + 16iz = 0$

10.5. $z^6 = -64i$

10.6. $z^3\bar{z} = 3i$

10.7. $\frac{z^4}{|z|} + 2 - 2i = 0$

11. Determine na forma $a + bi, a, b \in \mathbb{R}$ as raízes cúbicas dos números complexos:

11.1. $\frac{27}{i}$

11.2. $\frac{1+i}{1-i}$

11.3. $1 - i$

12. Sabe-se que z_1 é uma das raízes quartas de um complexo z.

12.1. Determine na forma $a + bi, a, b \in \mathbb{R}$ as outras raízes quartas de z de:

12.1.1. $z_1 = 1 + i$

12.1.2. $z_1 = 1 + 2i$

12.2. Determine, em cada uma das alíneas anteriores, o valor de z, na forma $a + bi, a, b \in \mathbb{R}$.

13. Considere o número complexo $z_1 = 81e^{i\frac{\pi}{4}}$.

13.1. Mostre que z_1 é solução da equação $iz = -\bar{z}$.

13.2. Determine a área do polígono cujas imagens são as raízes quartas de z_1.

14. Na figura está representado um pentágono regular. Sabe-se que o ponto A tem coordenadas $(4, 0)$.

14.1. Determine, na forma trigonométrica, os complexos cujos pontos afixos são os restantes vértices do pentágono.

14.2. Escreva uma equação em $\mathbb{C}$ cujas soluções sejam os números complexos encontrado na alínea anterior.

14.3. Seja z o número complexo que tem o ponto D como afixo. Indique, na forma trigonométrica, o número complexo w unitário, tal que $zw = 4$.

Soluções

1.

$$|z_2| = |-5\sqrt{3} + 5i| = \sqrt{25 \times 3 + 25} = \sqrt{100} = 10$$

$$\cos\theta = -\frac{5\sqrt{3}}{10} = -\frac{\sqrt{3}}{2} \text{ e } \sin\theta = \frac{5}{10} = \frac{1}{2}$$

$\frac{5\pi}{6}$ é um argumento de z_2 que pode ser escrito como $10e^{i\frac{5\pi}{6}}$

$$z_3 = 2i = 2e^{i\frac{\pi}{2}}$$

1.1.

$$z_1 z_3 = 2e^{-i\frac{2\pi}{5}} \times 2e^{i\frac{\pi}{2}} = 4e^{i\left(-\frac{2\pi}{5}+\frac{\pi}{2}\right)} = 4e^{i\frac{\pi}{10}}$$

1.2.

$$\overline{z_1} = 2e^{i\frac{2\pi}{5}}$$

$$z_2 \times \overline{z_1} = 10e^{i\frac{5\pi}{6}} \times 2e^{i\frac{2\pi}{5}} = 20e^{i\left(\frac{5\pi}{6}+\frac{2\pi}{5}\right)} = 20e^{i\frac{37\pi}{30}}$$

1.3.

$$\frac{z_1}{z_3} = \frac{2e^{-i\frac{2\pi}{5}}}{2e^{i\frac{\pi}{2}}} = e^{i\left(-\frac{2\pi}{5}-\frac{\pi}{2}\right)} = e^{-i\frac{9\pi}{10}}$$

1.4.

$$\frac{iz_1}{-z_2} = \frac{\left(e^{i\frac{\pi}{2}} \times 2e^{-i\frac{2\pi}{5}}\right)}{10e^{i\left(\frac{5\pi}{6}+\pi\right)}} = \frac{2e^{i\left(\frac{\pi}{2}-\frac{2\pi}{5}\right)}}{10e^{i\frac{11\pi}{6}}} = \frac{e^{i\frac{\pi}{10}}}{5e^{i\frac{11\pi}{6}}} = \frac{1}{5}e^{i\left(\frac{\pi}{10}-\frac{11\pi}{6}\right)} = \frac{1}{5}e^{-i\frac{26\pi}{15}}$$

1.5.

$$z_1^2 \times i^{22} = \left(2e^{-i\frac{2\pi}{5}}\right)^2 \times i^{4\times 5+2} = 4e^{-i\frac{4\pi}{5}} \times (-1) = 4e^{i\left(-\frac{4\pi}{5}+\pi\right)} = 4e^{i\frac{\pi}{5}}$$

2.

2.1.

$$z = e^{i\frac{2\pi}{3}} = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \vee$$

$$\vee\ z = e^{-i\frac{2\pi}{3}} = \cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$\left(a = -\frac{1}{2} \wedge b = \frac{\sqrt{3}}{2}\right) \vee \left(a = -\frac{1}{2} \wedge b = -\frac{\sqrt{3}}{2}\right)$$

2.2.

$$z = e^{i\frac{3\pi}{2}} = \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) = 0 - 1i \vee$$

$$\vee\ z = e^{-i\frac{3\pi}{2}} = \cos\left(-\frac{3\pi}{2}\right) + i\sin\left(-\frac{3\pi}{2}\right) = 0 + 1i$$

$$(a = 0 \wedge b = -1) \vee (a = 0 \wedge b = 1)$$

2.3.

$$z = e^{i\frac{5\pi}{4}} = \cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i \vee$$

$$\vee\ z = e^{-i\frac{5\pi}{4}} = \cos\left(-\frac{5\pi}{4}\right) + i\sin\left(-\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$$

$$\left(a = -\frac{\sqrt{2}}{2} \wedge b = -\frac{\sqrt{2}}{2}\right) \vee \left(a = -\frac{\sqrt{2}}{2} \wedge b = \frac{\sqrt{2}}{2}\right)$$

3.

$$A \to 1 \qquad B \to 3 \qquad C \to 2\sqrt{2}\,e^{i\frac{\pi}{4}}$$

$$A':\ 1 = e^{i0} \qquad z_{A'} = e^{i\left(0+\frac{2\pi}{3}\right)} = e^{i\frac{2\pi}{3}}$$

$$B':\ 3 = 3e^{i0} \qquad z_{B'} = 3e^{i\left(0+\frac{2\pi}{3}\right)} = 3e^{i\frac{2\pi}{3}}$$

$$C':\ 2\sqrt{2}\,e^{i\frac{\pi}{4}} \qquad z_{C'} = 2\sqrt{2}\,e^{i\left(\frac{\pi}{4}+\frac{2\pi}{3}\right)} = 2\sqrt{2}\,e^{i\frac{11\pi}{12}}$$

4.

4.1.

4.1.1.

$$iz = e^{i\frac{\pi}{2}} \times e^{i\theta} = e^{i\left(\theta+\frac{\pi}{2}\right)} \qquad \theta + \frac{\pi}{2}$$

4.1.2.

$$\frac{\bar{z}}{3i} = \frac{e^{-i\theta}}{3e^{i\frac{\pi}{2}}} = \frac{1}{2}e^{i\left(-\theta-\frac{\pi}{2}\right)} \qquad -\theta - \frac{\pi}{2}$$

4.1.3.

$$\left(\sqrt{2} - \sqrt{2}\,i\right)z$$

$$\left|\sqrt{2} - \sqrt{2}\,i\right| = \sqrt{2+2} = 2$$

$$\cos\theta = \frac{\sqrt{2}}{2} \text{ e } \sin\theta = -\frac{\sqrt{2}}{2} \qquad \theta = -\frac{\pi}{4}$$

$$\left(\sqrt{2} - \sqrt{2}\,i\right)z = 2e^{-i\frac{\pi}{4}} \times e^{i\theta} = 2e^{i\left(\theta-\frac{\pi}{4}\right)} \qquad \theta - \frac{\pi}{4}$$

4.2.

$$\theta = \frac{5\pi}{6} \qquad z = e^{i\frac{\pi}{2}}$$

$$|w| = \left|-\sqrt{3} - 1\right| = \sqrt{3+1} = \sqrt{4} = 2$$

$$\cos\theta = -\frac{\sqrt{3}}{2} \text{ e } \sin\theta = -\frac{1}{2} \qquad \theta = \frac{7\pi}{6}$$

$$w = 2e^{i\frac{7\pi}{6}}$$

$$\frac{7\pi}{6} - \frac{5\pi}{6} = \frac{2\pi}{6} = \frac{\pi}{3}$$

5.

5.1.

$$\left(2e^{i\frac{\pi}{4}}\right)^5 = 2^5 e^{i\frac{5\pi}{4}} = 32\left(\cos\frac{5\pi}{4} + \sin\frac{5\pi}{4}i\right) = 32\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = -16\sqrt{2} - 16\sqrt{2}\,i$$

5.2.

$$|\sqrt{3}-i|=\sqrt{3+1}=2$$

$$\cos\theta=\frac{\sqrt{3}}{2} \text{ e } \sin\theta=-\frac{1}{2}\ ,\ \theta=-\frac{\pi}{6}$$

$$\left(\sqrt{3}-i\right)^{10}=\left(2e^{-i\frac{\pi}{6}}\right)^{10}=2^{10}\times e^{-i\frac{10\pi}{6}}=2^{10}\times e^{-i\frac{5\pi}{3}}=2^{10}\times\left(\cos\left(-\frac{5\pi}{3}\right)+i\sin\left(-\frac{5\pi}{3}\right)\right)=$$

$$=1024\times\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=512+512\sqrt{3}\,i$$

5.3.

$$-1=\cos\pi+i\sin\pi=e^{i\pi}$$

6.

$$|1+i|=\sqrt{2} \qquad \cos\theta=\frac{\sqrt{2}}{2} \text{ e } \sin\theta=\frac{\sqrt{2}}{2}\ \rightarrow\ \theta=\frac{\pi}{4}$$

$$1+i=\sqrt{2}\,e^{i\frac{\pi}{4}}$$

$$|1-i|=\sqrt{2} \qquad \cos\theta=\frac{\sqrt{2}}{2} \text{ e } \sin\theta=-\frac{\sqrt{2}}{2}\ \rightarrow\ \theta=-\frac{\pi}{4}$$

$$1-i=\sqrt{2}\,e^{-i\frac{\pi}{4}}$$

$$\left|\frac{(1+i)^5}{i(1-i)^3}\right|=\left|\frac{\left(\sqrt{2}\,e^{i\frac{\pi}{4}}\right)^5}{e^{i\frac{\pi}{2}}\times\left(\sqrt{2}\,e^{-i\frac{\pi}{4}}\right)^3}\right|=\left|\frac{4\sqrt{2}\,e^{i\frac{5\pi}{4}}}{e^{i\frac{\pi}{2}}\times 2\sqrt{2}\,e^{-i\frac{3\pi}{4}}}\right|=\left|\frac{2e^{i\frac{5\pi}{4}}}{e^{-i\frac{\pi}{4}}}\right|=\left|2e^{i\frac{3\pi}{2}}\right|=2$$

7.

Considerando z na forma trigonométrica temos $z=|z|e^{i\theta}$, e $z^3=|z|^3e^{3i\theta}$

Como a imagem geométrica de z pertence ao primeiro quadrante, temos que $0<\theta<\frac{\pi}{2}$, e assim:

$$3\times 0<3\theta<3\times\frac{\pi}{2}\Leftrightarrow 0<3\theta<\frac{3\pi}{2}$$

Logo, $0<arg(z^3)<\frac{3\pi}{2}$, pelo que, a imagem geométrica de z^3 pode pertencer ao primeiro quadrante $\left(\text{se } 0<3\theta<\frac{\pi}{2}\right)$, ou ao segundo $\left(\text{se } \frac{\pi}{2}<3\theta<\pi\right)$, ou ao terceiro $\left(\text{se } \pi<3\theta<\frac{3\pi}{2}\right)$, mas nunca ao quarto quadrante.

8.

8.1.

$$|z_1|=\sqrt{2} \qquad \tan\theta=-1 \text{ e } \theta\in 2.^{\text{o}}\text{ Q}, \theta=\frac{3\pi}{4}$$

Na forma trigonométrica, $z_1=\sqrt{2}\,e^{i\frac{3\pi}{4}}$

$$\overline{z_1^4}=\overline{\left(\sqrt{2}\,e^{i\frac{3\pi}{4}}\right)^4}=\overline{4e^{i3\pi}}=4e^{-i3\pi}=-4$$

$$\frac{\overline{z_1^4}-2}{3i}=\frac{-4-2}{3i}=-\frac{6}{3i}=-\frac{2}{i}=2i$$, que é um imaginário puro

8.2.

$$z^3 z_1 = \frac{z}{\overline{z_2}} \Leftrightarrow z^2 \times z_1 = \frac{1}{\overline{z_2}} \Leftrightarrow z^2 = \frac{1}{z_1 \times \overline{z_2}} \Leftrightarrow$$

$$\Leftrightarrow z^2 = \frac{1}{\sqrt{2}\,e^{i\frac{3\pi}{4}} \times \sqrt{2}\,e^{-i\frac{\pi}{6}}} \Leftrightarrow z^2 = \frac{e^{i0}}{2e^{\frac{7\pi}{12}}} \Leftrightarrow z^2 = \frac{1}{2}e^{-i\frac{7\pi}{12}}$$

Seja $z = |z|e^{i\alpha}$ $\qquad z^2 = |z|^2 e^{i2\alpha}$

$$|z|^2 e^{i2\alpha} = \frac{1}{2}e^{i\frac{7\pi}{12}} \Leftrightarrow \begin{cases} |z| = \frac{\sqrt{2}}{2} \\ 2\alpha = \frac{7\pi}{12} \end{cases} \Leftrightarrow \begin{cases} |z| = \frac{\sqrt{2}}{2} \\ \alpha = \frac{7\pi}{24} \end{cases}$$

$$z = \frac{\sqrt{2}}{2}e^{-i\frac{7\pi}{24}}$$

9.

9.1.

$$(re^{i\theta})^2 = -2 \Leftrightarrow r^2 e^{i2\theta} = 2e^{-i\pi} \Leftrightarrow \begin{cases} r^2 = 2 \\ 2\theta = -\pi + 2k\pi, k \in \mathbb{Z} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} r = \sqrt{2}\,(r > 0) \\ \theta = -\frac{\pi}{2} + k\pi, k \in \mathbb{Z} \end{cases} \Leftrightarrow \begin{cases} r = \sqrt{2} \\ \theta = -\frac{\pi}{2}, (\theta \in]-\pi, \pi[) \end{cases}$$

9.2.

$$(re^{i\theta})^3 = -2 \Leftrightarrow r^3 e^{i3\theta} = 2e^{-i\pi} \Leftrightarrow \begin{cases} r^3 = 2 \\ 3\theta = -\pi + 2k\pi, k \in \mathbb{Z} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} r = \sqrt[3]{2} \\ \theta = -\frac{\pi}{3} + \frac{2k\pi}{3}, k \in \mathbb{Z} \end{cases} \Leftrightarrow \begin{cases} r = \sqrt[3]{2} \\ \theta = -\frac{\pi}{3}, (\theta \in]-\pi, \pi]) \end{cases}$$

9.3.

$$(re^{i\theta})^4 = -2 \Leftrightarrow r^4 e^{i4\theta} = 2e^{-i\pi} \Leftrightarrow \begin{cases} r^4 = 2 \\ 4\theta = -\pi + 2k\pi, k \in \mathbb{Z} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} r = \sqrt[4]{2} \\ \theta = -\frac{\pi}{4} + \frac{k\pi}{2}, k \in \mathbb{Z} \end{cases} \Leftrightarrow \begin{cases} r = \sqrt[4]{2} \\ \theta = -\frac{\pi}{4}, (\theta \in]-\pi, \pi]) \end{cases}$$

10.

10.1.

$$z^4 - 1 = 0 \Leftrightarrow z^4 = 1 \Leftrightarrow \left(|z|e^{i\alpha}\right)^4 = e^{i0} \Leftrightarrow |z|^4 \times e^{i4\alpha} = e^{i\pi} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} |z|^4 = 1 \\ 4\alpha = 2k\pi, k \in \mathbb{Z} \end{cases} \Leftrightarrow \begin{cases} |z| = 1 \\ \alpha = \frac{k\pi}{2}, k \in \mathbb{Z} \end{cases}$$

$$z_0 = e^{i \times 0} = 1\,,\ z_1 = e^{\frac{i\pi}{2}} = i\,,\ z_2 = e^{i\pi} = -1\,;\ z_3 = e^{\frac{i3\pi}{2}} = -i$$

$$\text{C.S.} = \{1, -1, i, -i\}$$

10.2.

$$z^3 = 27i \Leftrightarrow \left(|z|e^{i\alpha}\right)^3 = 27e^{\frac{i\pi}{2}} \Leftrightarrow |z|^3 e^{i3\alpha} = 27e^{\frac{i\pi}{2}} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} |z|^3 = 27 \\ 3\alpha = \frac{\pi}{2} + 2k\pi, k \in \mathbb{Z} \end{cases} \Leftrightarrow \begin{cases} |z| = 3 \\ \alpha = \frac{\pi}{6} + \frac{2k\pi}{3}, k \in \mathbb{Z} \end{cases}$$

$$z_0 = 3e^{\frac{i\pi}{6}}, \ z_1 = 3e^{\frac{i5\pi}{6}}, \ z_2 = 3e^{\frac{i3\pi}{2}}$$

$$\text{C.S.} = \left\{3e^{\frac{i\pi}{6}}, 3e^{\frac{i5\pi}{6}}, 3e^{\frac{i3\pi}{2}}\right\}$$

10.3.

$$z^2 + 16i = 0 \Leftrightarrow z^2 = -16i \Leftrightarrow \left(|z|e^{i\alpha}\right)^2 = 16e^{\frac{i3\pi}{2}} \Leftrightarrow |z|^2 e^{i2\alpha} = 16e^{\frac{i3\pi}{2}} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} |z|^2 = 16 \\ 2\alpha = \frac{3\pi}{2} + 2k\pi, k \in \mathbb{Z} \end{cases} \Leftrightarrow \begin{cases} |z| = 4 \\ \alpha = \frac{3\pi}{4} + k\pi, k \in \mathbb{Z} \end{cases}$$

$$z_0 = 4e^{\frac{i3\pi}{4}}, \ z_1 = 4e^{\frac{i7\pi}{4}}$$

$$\text{C.S.} = \left\{e^{\frac{i3\pi}{4}}, e^{\frac{i7\pi}{4}}\right\}$$

10.4.

$$z^3 + 16iz = 0 \Leftrightarrow z(z^2 + 16i) = 0 \Leftrightarrow z = 0 \lor z^2 + 16i = 0 \Leftrightarrow z = 0 \lor z^2 = -16i$$

Seja $z = |z|e^{i\alpha}$, $z = 0 \lor |z|^2 e^{i2\alpha} = 16e^{i\left(\frac{3\pi}{2}\right)} \Leftrightarrow$

$$\Leftrightarrow z = 0 \lor \begin{cases} |z| = 4 \\ 2\alpha = \frac{3\pi}{2} + 2k\pi, k \in \mathbb{Z} \end{cases} \Leftrightarrow z = 0 \lor \begin{cases} |z| = 4 \\ \alpha = \frac{3\pi}{4} + k\pi, k \in \mathbb{Z} \end{cases}$$

$$z = 0 \lor z = 4e^{i\frac{3\pi}{4}} \lor z = 4e^{\frac{7\pi}{4}}$$

$$\text{C.S.} = \left\{0, 4e^{i\frac{3\pi}{4}}, 4e^{i\frac{7\pi}{4}}\right\}$$

10.5.

$$z^6 = -64i$$

Seja $z = |z|e^{i\alpha}$,

$$|z|^6 e^{i6\alpha} = 64e^{i\frac{3\pi}{2}} \Leftrightarrow \begin{cases} |z|^6 = 64 \\ 6\alpha = \frac{3\pi}{2} + 2k\pi, k \in \{0, 1, 2, 3, 4, 5\} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} |z| = 2 \\ \alpha = \frac{\pi}{4} + \frac{k\pi}{3}, k \in \{0, 1, 2, 3, 4, 5\} \end{cases}$$

$$z = 2e^{\frac{i\pi}{4}} \lor z = 2e^{\frac{i7\pi}{12}} \lor z = 2e^{\frac{i11\pi}{12}} \lor z = 2e^{\frac{i5\pi}{4}} \lor z = 2e^{\frac{i19\pi}{12}} \lor z = 2e^{\frac{i23\pi}{12}}$$

$$\text{C.S.} = \left\{2e^{\frac{i\pi}{4}}, 2e^{\frac{i7\pi}{12}}, 2e^{\frac{i11\pi}{12}}, 2e^{\frac{i5\pi}{4}}, 2e^{\frac{i19\pi}{12}}, 2e^{\frac{i23\pi}{12}}\right\}$$

10.6.

$z^3\overline{z} = 3i$

Seja $z = |z|e^{i\alpha}$, $\quad \overline{z} = |z|\, e^{-i\alpha}$

$z^3\overline{z} = 3i \Leftrightarrow |z|^3 e^{i3\alpha} \times |z|\, e^{-i\alpha} = 3e^{\frac{i\pi}{2}} \Leftrightarrow |z|^4 e^{i2\alpha} = 3e^{\frac{i\pi}{2}} \Leftrightarrow$

$$\Leftrightarrow \begin{cases} |z|^4 = 3 \\ 2\alpha = \frac{\pi}{2} + 2k\pi, k \in \mathbb{Z} \end{cases} \Leftrightarrow \begin{cases} |z| = \sqrt[4]{3} \\ \alpha = \frac{\pi}{4} + k\pi, k \in \mathbb{Z} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow z = \sqrt[4]{3}\, e^{\frac{i\pi}{4}} \vee z = \sqrt[4]{3}\, e^{\frac{i5\pi}{4}}$$

$$\text{C.S.} = \left\{ \sqrt[4]{3}\, e^{\frac{i\pi}{4}}, \sqrt[4]{3}\, e^{\frac{i5\pi}{4}} \right\}$$

10.7.

$\frac{z^4}{|z|} + 2 - 2i = 0 \Leftrightarrow \frac{z^4}{|z|} = -2 + 2i$

$|-2 + 2i| = \sqrt{4+4} = 2\sqrt{2}$

$\sin\theta = \frac{\sqrt{2}}{2} \qquad \cos\theta = -\frac{\sqrt{2}}{2}, \qquad$ logo, $\theta = \frac{3\pi}{4}$.

Na forma trigonométrica, $2 - 2i$ escreve-se como $2\sqrt{2}\, e^{\frac{i3\pi}{4}}$.

Seja $z = |z|e^{i\alpha}$

$$\frac{|z|^4 e^{i4}}{|z|} = 2\sqrt{2}\, e^{\frac{i3\pi}{4}} \Leftrightarrow \begin{cases} |z|^3 = 2\sqrt{2} \\ 4\alpha = \frac{3\pi}{4} + 2k\pi, k \in \mathbb{Z} \end{cases} \Leftrightarrow \begin{cases} |z| = \sqrt[3]{\sqrt{8}} = \sqrt{2} \\ \alpha = \frac{3\pi}{16} + \frac{k\pi}{2}, k \in \mathbb{Z} \end{cases}$$

$$z = \sqrt{2}\, e^{\frac{i3\pi}{16}} \vee z = \sqrt{2}\, e^{\frac{i11\pi}{16}} \vee z = \sqrt{2}\, e^{\frac{i19\pi}{16}} \vee z = \sqrt{2}\, e^{\frac{i27\pi}{16}}$$

$$\text{C.S.} = \left\{ \sqrt{2}\, e^{\frac{i3\pi}{16}}, \sqrt{2}\, e^{\frac{i11\pi}{16}}, \sqrt{2}\, e^{\frac{i19\pi}{16}}, \sqrt{2}\, e^{\frac{i27\pi}{16}} \right\}$$

11.

11.1.

$\frac{27}{i} = -27i = 27e^{\frac{i3\pi}{2}}$

As raízes cúbicas de $27e^{\frac{i3\pi}{2}}$ são da forma $z_k = \sqrt[3]{27}\, e^{i\left(\frac{\pi}{2} + \frac{2k\pi}{3}\right)}$, $k = 0, 1, 2$

$z_0 = 3e^{\frac{i\pi}{2}} = 3i$

$$z_1 = 3e^{\frac{i11\pi}{6}} = 3\left(\cos\left(\frac{11\pi}{6}\right) + i\sin\pi\left(\frac{11\pi}{6}\right)\right) = 3 \times \left(\frac{\sqrt{3}}{2} + i \times \left(-\frac{1}{2}\right)\right) = \frac{3\sqrt{3}}{2} - \frac{3}{2}i$$

$$z_2 = 3e^{\frac{i7\pi}{6}} = 3\left(\cos\frac{7\pi}{6} + i\sin\left(\frac{7\pi}{6}\right)\right) = 3 \times \left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = -\frac{3\sqrt{2}}{2} - \frac{3}{2}i$$

11.2.

$$\frac{1+i}{1-i}=\frac{(1+i)(1-i)}{1+1}=\frac{1+2i-1}{2}=i=e^{\frac{i\pi}{2}}$$

As raízes cúbicas de $e^{\frac{i\pi}{2}}$ são da forma $z_k=e^{i\left(\frac{\pi}{6}+\frac{2k\pi}{3}\right)},\ k=0,1,2$

$$z_0=e^{\frac{i\pi}{6}}=\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}=\frac{\sqrt{3}}{2}+\frac{1}{2}i$$

$$z_1=e^{\frac{i5\pi}{6}}=\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}=-\frac{\sqrt{3}}{2}+\frac{1}{2}i$$

$$z_2=e^{\frac{i3\pi}{2}}=-i$$

11.3.

$$|1-i|=\sqrt{2}\qquad \cos\theta=\frac{\sqrt{2}}{2}\ \text{e}\ \sin\theta=-\frac{\sqrt{2}}{2},\ \text{logo},\ \theta=-\frac{\pi}{4}.$$

Assim, na forma trigonométrica: $1-i=\sqrt{2}\,e^{-\frac{i\pi}{4}}$

As raízes cúbicas de $\sqrt{2}\,e^{-\frac{i\pi}{4}}$ são da forma $\sqrt[6]{2}\,e^{i\left(-\frac{\pi}{12}+\frac{2k\pi}{3}\right)}$

$$z_0=\sqrt[6]{2}\,e^{-i\frac{\pi}{12}}\qquad z_1=\sqrt[6]{2}\,e^{i\frac{7\pi}{12}}\qquad z_2=\sqrt[6]{2}\,e^{i\frac{5\pi}{4}}$$

$$z_0=\sqrt[6]{2}\,e^{-i\frac{\pi}{12}}=\sqrt[6]{2}\left(\cos\left(-\frac{\pi}{12}\right)+i\sin\left(-\frac{\pi}{12}\right)\right)=$$

$$=\sqrt[6]{2}\left(\frac{\sqrt{2}+\sqrt{6}}{4}+\frac{\sqrt{2}-\sqrt{6}}{4}i\right)=\sqrt[6]{2}\frac{\sqrt{2}+\sqrt{6}}{4}+\sqrt[6]{2}\frac{\sqrt{2}-\sqrt{6}}{4}i$$

$$z_1=\sqrt[6]{2}\,e^{i\frac{7\pi}{12}}=\sqrt[6]{2}\left(\cos\left(\frac{7\pi}{12}\right)+i\sin\left(\frac{7\pi}{12}\right)\right)=$$

$$=\sqrt[6]{2}\left(\frac{\sqrt{2}-\sqrt{6}}{4}+\frac{\sqrt{2}+\sqrt{6}}{4}i\right)=\sqrt[6]{2}\frac{\sqrt{2}-\sqrt{6}}{4}+\sqrt[6]{2}\frac{\sqrt{2}+\sqrt{6}}{4}i$$

$$z_2=\sqrt[6]{2}\,e^{i\frac{5\pi}{4}}=\sqrt[6]{2}\left(\cos\left(\frac{5\pi}{4}\right)+i\sin\left(\frac{5\pi}{4}\right)\right)=$$

$$=\sqrt[6]{2}\left(-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)=-\frac{\sqrt[3]{2^2}}{2}-\frac{\sqrt[3]{2^2}}{2}i=-\frac{\sqrt[3]{4}}{2}-\frac{\sqrt[3]{4}}{2}i$$

Cálculos auxiliares:

$$\cos\left(-\frac{\pi}{12}\right)=\cos\left(\frac{\pi}{4}-\frac{\pi}{3}\right)=\cos\left(\frac{\pi}{4}\right)\times\cos\left(\frac{\pi}{3}\right)+\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{3}\right)=$$

$$=\frac{\sqrt{2}}{2}\times\frac{1}{2}+\frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2}=\frac{\sqrt{2}+\sqrt{6}}{4}$$

$$\sin\left(-\frac{\pi}{12}\right)=\sin\left(\frac{\pi}{4}-\frac{\pi}{3}\right)=\sin\frac{\pi}{4}\cos\frac{\pi}{3}-\cos\frac{\pi}{4}\sin\frac{\pi}{3}=$$

$$=\frac{\sqrt{2}}{2}\times\frac{1}{2}-\frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2}=\frac{\sqrt{2}-\sqrt{6}}{4}$$

$$\cos\left(\frac{7\pi}{12}\right) = \cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\sin\frac{\pi}{4} =$$

$$= \frac{1}{2} \times \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} = \frac{\sqrt{2} - \sqrt{6}}{4}$$

$$\sin\left(\frac{7\pi}{12}\right) = \sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right) = \sin\frac{\pi}{3}\cos\frac{\pi}{4} + \cos\frac{\pi}{3}\sin\frac{\pi}{4} =$$

$$= \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} + \frac{1}{2} \times \frac{\sqrt{2}}{2} = \frac{\sqrt{6} + \sqrt{2}}{4}$$

12.

12.1.

12.1.1.

$(z_1)^4 = (1 + i)^4 = ((1 + i)^2)^2 = (1 + 2i - 1)^2 = -4 = 4e^{-i\pi}$

As raízes quartas de $4e^{-i\pi}$ são da forma $\sqrt[4]{4}\, e^{i\left(-\frac{\pi}{4} + \frac{2k\pi}{4}\right)}$ $k = 0, 1, 2, 3$

$z_0 = \sqrt{2}\, e^{\frac{-i\pi}{4}} = 1 - i$, $z_1 = \sqrt{2}\, e^{\frac{i\pi}{4}} = 1 + i$,

$z_2 = \sqrt{2}\, e^{\frac{i3\pi}{4}} = -1 + i$, $z_3 = \sqrt{2}\, e^{\frac{i5\pi}{4}} = -1 - i$

As outras raízes quartas de z são z_1 , z_2 , z_3 .

12.1.2.

$(z_1)^4 = (1 + 2i)^4 = ((1 + 2i)^2)^2 = (1 + 4i - 4)^2 = (-3 + 4i)^2 =$
$= 9 - 24i - 16 = -7 - 24i$

$|-7 - 24i| = \sqrt{49 + 576} = \sqrt{625} = 25$

As raízes quartas de z são números complexos cujos afixos são vértices de quadrado inscrito numa circunferência de raio $\sqrt{5}$.

Podemos obter os restantes vértices aplicando rotações sucessivas de $\frac{\pi}{2}$ rad e centro na origem, ou seja, o que corresponde a multiplicar z_1 por $e^{\frac{i\pi}{2}} = i$

Assim:

$z_2 = (z_1 \times i) = (1 + 2i)i = -2 + i$

$z_3 = (z_2 \times i) = (-2 + i) \times i = -1 - 2i$

$z_0 = (z_3 \times i) = (-1 - 2i) \times i = 2 - i$

12.2.

$z = -4$

$z = -7 - 24i$

13.

$z_1 = 81e^{i\frac{\pi}{4}}$ $\qquad \overline{z_1} = 81e^{-i\frac{\pi}{4}}$ $\qquad -\overline{z_1} = 81e^{i\left(-\frac{\pi}{4} + \pi\right)} = 81e^{i\left(\frac{3\pi}{4}\right)}$

13.1.

$$iz_1 = e^{\frac{i\pi}{2}} \times 81e^{i\frac{\pi}{4}} = 81e^{i\left(\frac{3\pi}{4}\right)} = -\overline{z_1}$$

13.2.

As raízes quartas de z_1 são da forma $z_k = \sqrt[4]{81}e^{i\left(\frac{\pi}{16}+\frac{2k\pi}{4}\right)}$,
$k = 0, 1, 2, 3$

$z_0 = 3e^{i\frac{\pi}{16}}$, $z_1 = 3e^{i\frac{9\pi}{16}}$, $z_2 = 3e^{i\frac{17\pi}{16}}$, $z_3 = 3e^{i\frac{25\pi}{16}}$

Trata-se de um quadrado cuja distância de cada vértice à origem do referencial é de 3 unidades

$l^2 = 3^2 + 3^2 \Leftrightarrow l^2 = 18 \Leftrightarrow A = 18$ u.a.

14.

14.1.

$z_1 = 4e^{i\times 0}$ é o complexo cujo afixo é o ponto A .

Como o pentágono é regular, $A\widehat{O}B = B\widehat{O}C = C\widehat{O}D = D\widehat{O}E = E\widehat{O}A = \frac{2\pi}{5}$

Então, $z_2 = z_1e^{\frac{i2\pi}{5}}$, $z_3 = z_1e^{\frac{i4\pi}{5}}$, $z_4 = z_1e^{\frac{i6\pi}{5}}$ e $z_5 = z_1e^{\frac{i8\pi}{5}}$
são os complexos que têm como afixos os vértices B , C , D e E , respetivamente.

$z_2 = 4e^{\frac{i2\pi}{5}}$, $z_3 = 4e^{\frac{i4\pi}{5}}$, $z_4 = 4e^{\frac{i6\pi}{5}}$ e $z_5 = 4e^{\frac{i8\pi}{5}}$

14.2. $z^5 = 1024$

14.3.

$z = 4e^{\frac{6\pi}{5}}$

Seja $w = e^{i\alpha}$

$zw = 4 \Leftrightarrow 4e^{\frac{i6\pi}{5}} \times e^{i\alpha} = 4 \Leftrightarrow e^{i\left(\frac{6\pi}{5}+\alpha\right)} = 1 \Leftrightarrow$

$\Leftrightarrow \frac{6\pi}{5} + \alpha = 2k\pi$, $k \in \mathbb{Z} \Leftrightarrow \alpha = -\frac{6\pi}{5} + 2k\pi$, $k \in \mathbb{Z}$

Assim: $w = e^{-\frac{i6\pi}{5}}$