Nome do aluno | Nº | Data / / 20

# Propriedades algébricas dos logaritmos

**1.** Sabendo que $\log_2 a = 5$, determine:

**1.1.** $\log_2(\sqrt{2}a)$

**1.2.** $\log_2\left(\frac{\sqrt{8}}{0{,}5a}\right)$

**1.3.** $\log_2 6 + \log_2\left(\frac{1}{3a}\right)$

**2.** Sabendo que $\log_2 9 = x$, represente em função de $x$:

**2.1.** $\log_2 6$

**2.2.** $\log_2 64 - \log_2 27$

**2.3.** $\log_2 \frac{\sqrt{3}}{4}$

**3.** Dados $x, y \in \mathbb{R}^+, x > y$, transforme num único logaritmo:

**3.1.** $-\log_3 x + \frac{1}{2}\log_3 y$

**3.2.** $\ln(x^2 - y^2) - 2\ln\left(\sqrt{x-y}\right)$

**4.** Utilizando a propriedade dos logaritmos de mudança de base, determine:

$$\log_7 32 \times \log_2 \sqrt{7}$$

**5.** Sejam $x, y \in \mathbb{R}^+$, $\log x = 2$ e $\log y = 3$. Determine:

**5.1.** $\log\left(\frac{x}{y}\right)$

**5.2.** $\log(x^2 y)$

**5.3.** $\log \sqrt[3]{\frac{\sqrt{y}}{100}}$

**5.4.** $\log_{100}(xy)^2$

**6.** Resolva as seguintes equações indicando o conjunto solução:

**6.1.** $\log_4(1-x) = 2$

**6.2.** $\log_{\frac{1}{3}}(5x-1) = \log_{\frac{1}{3}}(3x+7)$

**6.3.** $\log_2(x-3) + \log_2(x+3) = 4$

**6.4.** $\ln x - 2(\ln x)^2 = 0$

**6.5.** $\log_2\left(\frac{x-2}{2x-7}\right) = 3 - \log_2 x$

**6.6.** $\log_3(x-2) = 1 + \log_9(6-x)$

**7.** Seja $f$ a função definida em $\mathbb{R}^+$ por:

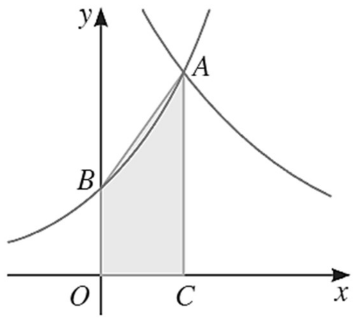
$$f(x) = \log_2(x^2) - \log_3\left(\frac{x}{9}\right)$$

**7.1.** Mostre que $f(x) = 2 + \log_3 x$ para qualquer $x \in \mathbb{R}^+$.

**7.2.** Determine a área do triângulo cujos vértices são a origem do referencial, o ponto do gráfico de $f$ cuja ordenada é 1 e o ponto correspondente ao zero de $f$.

**8.** Na figura estão representados, em referencial o.n. $Oxy$, um trapézio $[ABOC]$ e partes dos gráficos das funções, de domínio $\mathbb{R}$, definidas por:

$$f(x) = 2^{2-x} \qquad e \qquad g(x) = 3^x$$



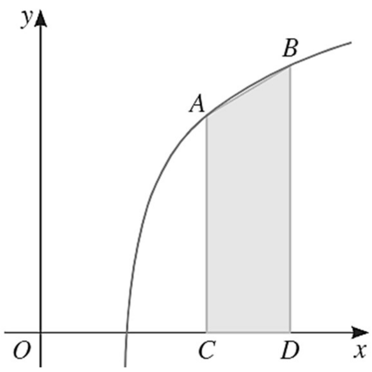

Como a figura sugere:

- O ponto $O$ é a origem do referencial;
- O ponto $A$ é a interseção dos gráficos de $f$ e $g$;
- O ponto $B$ é a interseção do gráfico de $g$ com o eixo das ordenadas;
- O ponto $C$ tem a mesma abcissa de $A$ e pertence ao eixo das abcissas.

Determine a área do trapézio $[ABOC]$, aproximada às centésimas.

**9.** No referencial da figura estão representados um trapézio $[ABCD]$ e parte do gráfico da função $f$ definida analiticamente por:

$$f(x) = 2 + \ln(x - e)$$




Sabe-se que:

- $A$ é o ponto do gráfico de $f$ de abcissa $2e$;
- $B$ é o ponto do gráfico de $f$ de ordenada $\ln(2e^3)$;
- Os pontos $C$ e $D$ pertencem ao eixo das abcissas.

Mostre que a área do trapézio $[ABCD]$ pode ser dada por: $e\ln\left(\sqrt{2}e^3\right)$.

**10.** Resolva, em $\mathbb{R}$, as seguintes condições:

**10.1.** $\log_3(x+2) \leq 1$

**10.2.** $\log_2(\log x) > -1$

**10.3.** $\log_3 x + \log_3(x+6) < 3$

**10.4.** $(\ln x)^2 + \ln x > 0$

**10.5.** $\log_{\frac{1}{5}}(x^2) \geq \log_{\frac{1}{5}} 4$

**10.6.** $\log_2(x+1)^2 < 0$

**11.** Apresente, em $\mathbb{R}$, o conjunto solução da condição:

$$\frac{\log x}{1 + \log x} \leq 2$$

**Soluções**

1.

1.1.

$$\log_2(\sqrt{2}a) = \log_2\sqrt{2} + \log_2 a = \frac{1}{2} + 5 = \frac{11}{2}$$

1.2.

$$\log_2\left(\frac{\sqrt{8}}{0{,}5a}\right) = \log_2(\sqrt{8}) - \log_2(0{,}5a) = \log_2(2^3)^{\frac{1}{2}} - \log_2 0{,}5 - \log_2 a =$$

$$= \log_2 2^{\frac{3}{2}} - \log_2 2^{-1} - \log_2 a = \frac{3}{2} + 1 - 5 = -\frac{5}{2}$$

1.3.

$$\log_2 6 + \log_2\left(\frac{1}{3a}\right) = \log_2\left(\frac{6}{3a}\right) = \log_2\left(\frac{2}{a}\right) = \log_2 2 - \log_2 a = 1 - 5 = -4$$

2.

2.1.

$$\log_2 6 = \log_2(2 \times 3) = \log_2 2 - \log_2 3 = 1 + \log_2 9^{\frac{1}{2}} =$$

$$= 1 + \frac{1}{2} \times \log_2 9 = 1 + \frac{1}{2}x$$

2.2.

$$\log_2 64 - \log_2 27 = \log_2 2^6 - \log_2(9 \times 3) = 6 - \log_2 9 - \log_2 3 =$$

$$= 6 - x - \log_2 9^{\frac{1}{2}} = 6 - x - \frac{1}{2}x = 6 - \frac{3}{2}x$$

2.3.

$$\log_2\frac{\sqrt{3}}{4} = \log_2\sqrt{3} - \log_2 4 = \log_2 3^{\frac{1}{2}} - \log_2 2^2 = \log_2 9^{\frac{1}{4}} - 2 =$$

$$= \frac{1}{4} \times \log_2 9 - 2 = \frac{1}{4}x - 2$$

3.

3.1.

$$-\log_3 x + \frac{1}{2}\log_3 y = \log_3\left(\frac{1}{x}\right) + \log_3\sqrt{y} = \log_3\left(\frac{\sqrt{y}}{x}\right)$$

3.2.

$$\ln(x^2 - y^2) - 2\ln\sqrt{x-y} = \ln(x^2 - y^2) - \ln\left(\sqrt{x-y}\right)^2 =$$

$$= \ln\left(\frac{x^2 - y^2}{x - y}\right) = \ln\left(\frac{(x-y)(x+y)}{x-y}\right) = \ln(x+y)$$

4.

$$\log_7 32 \times \log_2\sqrt{7} = \frac{\log_2 32}{\log_2 7} \times \log_2 7^{\frac{1}{2}} = \frac{\log_2 2^5}{\log_2 7} \times \frac{1}{2}\log_2 7 = 5 \times \frac{1}{2} = \frac{5}{2}$$

5.

5.1.

$$\log\left(\frac{x}{y}\right) = \log x - \log y = 2 - 3 = -1$$

5.2.

$$\log(x^2 y) = \log x^2 + \log y = 2\log x + \log y = 2 \times 2 + 3 = 7$$

**5.3.** D

$$\log\sqrt[3]{\frac{\sqrt{y}}{100}} = \log\left(\frac{\sqrt{y}}{100}\right)^{\frac{1}{3}} = \frac{1}{3}\log\left(\frac{\sqrt{y}}{100}\right) = \frac{1}{3}\left(\log y^{\frac{1}{2}} - \log 10^2\right) =$$

$$= \frac{1}{3} \times \left(\frac{1}{2}\log y - 2\right) = \frac{1}{6} \times 3 - \frac{2}{3} = -\frac{1}{6}$$

**5.4.**

$$\log_{100}(xy)^2 = 2\log_{100}xy = 2 \times (\log_{100}x + \log_{100}y) =$$

$$= 2 \times \frac{\log x}{\log 100} + 2 \times \frac{\log y}{\log 100} = 2 \times \frac{2}{2} + 2 \times \frac{3}{2} = 5$$

**6.**

**6.1.**

$\log_4(1-x) = 2$

$D = \{x \in \mathbb{R}: 1 - x > 0\} = ]-\infty, 1[$

$\log_4(1-x) = 2 \Leftrightarrow 1 - x = 4^2 \Leftrightarrow 1 - x = 16 \Leftrightarrow x = -15 \in D$

C.S. $= \{-15\}$

**6.2.**

$\log_{\frac{1}{3}}(5x-1) = \log_{\frac{1}{3}}(3x+7)$

$D = \{x \in \mathbb{R}: 5x - 1 > 0 \wedge 3x + 7 > 0\} = \left]\frac{1}{5}; +\infty\right[ \cap \left]\frac{7}{3}; +\infty\right[ = \left]\frac{1}{5}; +\infty\right[$

$5x - 1 = 3x + 7 \Leftrightarrow 2x = 8 \Leftrightarrow x = 4 \in D$

C.S. $= \{4\}$

**6.3.**

$\log_2(x-3) + \log_2(x+3) = 4$

$D = \{x \in \mathbb{R}: x - 3 > 0 \wedge x + 3 > 0\} = ]3, +\infty[ \cap ]-3, +\infty[ = ]3, +\infty[$

$\log_2((x-3)(x+3)) = \log_2 2^4 \Leftrightarrow x^2 - 9 = 16 \Leftrightarrow x^2 = 25 \Leftrightarrow x = \underset{(\notin D)}{-5} \vee x = 5$

C.S. $= \{5\}$

**6.4.**

$\ln x - 2(\ln x)^2 = 0$

$D = \{x \in \mathbb{R}: x > 0\} = \mathbb{R}^+$

$\ln x - 2(\ln x)^2 = 0 \Leftrightarrow \ln x(1 - 2\ln x) = 0 \Leftrightarrow \ln x = 0 \vee 2\ln x = 1 \Leftrightarrow$

$\Leftrightarrow \ln x = 0 \vee \ln x = \frac{1}{2} \Leftrightarrow x = 1 \vee x = e^{\frac{1}{2}}$

C.S. $= \{1, \sqrt{e}\}$

**6.5.** K

$\log_2\left(\frac{x-2}{2x-7}\right) = 3 - \log_2 x$

$D = \left\{x \in \mathbb{R}: \frac{x-2}{2x-7} > 0 \wedge x > 0\right\} = \left(]-\infty, 2[ \cup \left]\frac{7}{2}, +\infty\right[\right) \cap \mathbb{R}^+ = ]0, 2[ \cup \left]\frac{7}{2}, +\infty\right[$

$x - 2 = 0 \Leftrightarrow x = 2$

$2x - 7 = 0 \Leftrightarrow x = \frac{7}{2}$

| | $-\infty$ | 2 | | $\frac{7}{2}$ | $+\infty$ |
|---|---|---|---|---|---|
| $x-2$ | $-$ | 0 | $+$ | $+$ | $+$ |
| $2x-7$ | $-$ | $-$ | $-$ | 0 | $+$ |
| $\frac{x-2}{2x-7}$ | $+$ | 0 | $-$ | n.d. | $+$ |

$$\frac{x-2}{2x-7} > 0 \Leftrightarrow x \in \left]0, 2\right[ \cup \left]\frac{7}{2}, +\infty\right[$$

$$\log_2\left(\frac{x-2}{2x-7}\right) = \log_2 2^3 - \log_2 x \Leftrightarrow \log_2\left(\frac{x-2}{2x-7}\right) + \log_2 x =$$

$$= \log_2 8 \Leftrightarrow \log_2\left(\frac{x(x-2)}{2x-7}\right) = \log_2 8 \Leftrightarrow \frac{x(x-2)}{2x-7} = 8 \Leftrightarrow$$

$$\Leftrightarrow \frac{x(x-2)}{2x-7} - \frac{8(2x-7)}{2x-7} = 0 \Leftrightarrow \frac{x^2 - 2x - 16x + 56}{2x-7} = 0 \Leftrightarrow$$

$$\Leftrightarrow \frac{x^2 - 18x + 56}{2x-7} = 0 \Leftrightarrow x^2 - 18x + 56 = 0 \wedge 2x - 7 \neq 0 \Leftrightarrow$$

$$\Leftrightarrow x = \frac{18 \pm \sqrt{18^2 - 4 \times 1 \times 56}}{2} \Leftrightarrow x \neq \frac{7}{2} \Leftrightarrow x = 4 \vee x = 14 \wedge x \neq \frac{7}{2}$$

C.S. $= \{4, 14\}$

**6.6.**

$$\log_3(x-2) = 1 + \log_9(6-x)$$

$$D = \{x \in \mathbb{R}: x - 2 > 0 \wedge 6 - x > 0\} = \left]2, 6\right[$$

$$\log_3(x-2) = \log_3 3 + \frac{\log_3(6-x)}{\log_3 9} \Leftrightarrow$$

$$\Leftrightarrow \log_3(x-2) - \log_3 3 = \frac{1}{2}\log_3(6-x) \Leftrightarrow$$

$$2\log_3\left(\frac{x-2}{3}\right) = \log_3(6-x) \Leftrightarrow \log_3\left(\frac{x-2}{3}\right)^2 = \log_3(6-x) \Leftrightarrow$$

$$\Leftrightarrow \left(\frac{x-2}{3}\right)^2 = 6 - x \Leftrightarrow \frac{x^2 - 4x + 4}{9} = 6 - x \Leftrightarrow$$

$$\Leftrightarrow x^2 - 4x + 4 = 54 - 9x \Leftrightarrow x^2 + 5x - 50 = 0 \Leftrightarrow$$

$$\Leftrightarrow x = \frac{-5 \pm \sqrt{25 - 4 \times 1 \times (-50)}}{2} \Leftrightarrow x = \underset{(\notin D)}{-10} \vee x = 5$$

C.S. $= \{5\}$

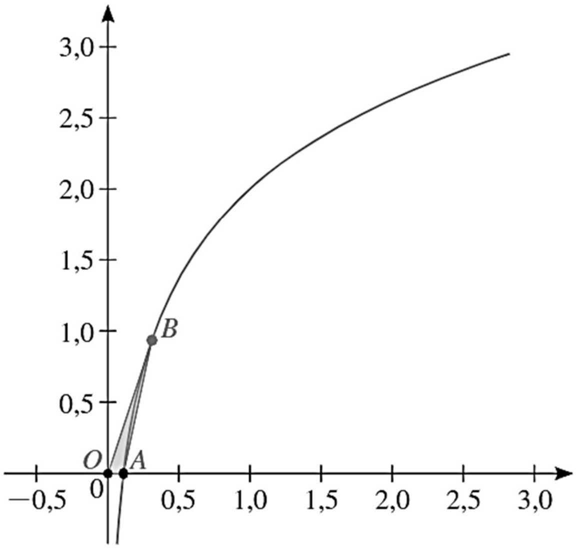
**7.**

**7.1.**

$$f(x) = \log_3(x^2) - \log_3\left(\frac{x}{9}\right) = \log_3(x^2) - \log_3 x + \log_3 9 = \log_3\left(\frac{x^2}{x}\right) + 2 = 2 + \log_3 x$$

**7.2.**

$$f(x) = 0 \Leftrightarrow 2 + \log_3 x = 0 \Leftrightarrow \log_3 x = -2 \Leftrightarrow \log_3 x = \log_3 3^{-2} \quad \Leftrightarrow x = \frac{1}{9} \rightarrow \text{zero de } f$$




$$A = \frac{\frac{1}{9} \times 1}{2} = \frac{1}{18}$$

**8.**

$\overline{OB} = 1$

Cálculo da abcissa de $A$ :

$2^{2-x} = 3^x \Leftrightarrow \log 2^{2-x} = \log 3^x \Leftrightarrow (2-x)\log 2 = x \log 3 \Leftrightarrow x = \dfrac{\log 4}{\log 6} = \log_6 4$

$\overline{OC} = \log_6 4$

$\overline{AC} = 3^{\log_6 4}$

A área do trapézio é: $\dfrac{3^{\log_6 4} + 1}{2} \times \log_6 4 \approx 1{,}29$ .

**9.**

$f(2e) = 2 + \ln(2e - e) = 2 + \ln e = 2 + 1 = 3 \qquad A = (2e, 3)$

$f(x) = \ln(2e^3) \Leftrightarrow 2 + \ln(x - e) = \ln(2e^3) \Leftrightarrow \ln e^2 + \ln(x - e) = \ln(2e^3) \Leftrightarrow$

$\Leftrightarrow \ln(e^2(x - e)) = \ln(2e^3) \Leftrightarrow e^2(x - e) = 2e^3 \Leftrightarrow x - e = 2e \Leftrightarrow x = 3e$

$B = (3e, \ln(2e^3))$

$$A_{\text{trapézio}} = \frac{\overline{AC} + \overline{BD}}{2} \times \overline{CD} = \frac{3 + \ln(2e^3)}{2} \times (3e - 2e) =$$

$$= \frac{3 + \ln(2e^3)}{2} \times e = \left(\frac{3}{2} + \frac{1}{2}\ln(2e^3)\right) \times e =$$

$$= \left(\ln e^{\frac{3}{2}} + \ln(2e^3)^{\frac{1}{2}}\right) \times e = \ln\left(e^{\frac{3}{2}} \times \sqrt{2}\, e^{\frac{3}{2}}\right) \times e =$$

$$= \ln\left(\sqrt{2}\, e^3\right) \times e = e \ln\left(\sqrt{2}\, e^3\right)$$

**10.**

**10.1.**

$\log_3(x+2) \leqslant 1$

$D = \{x \in \mathbb{R}: x+2>0\} = ]-2, +\infty[$

$\log_3(x+2) \leqslant 1 \Leftrightarrow \log_3(x+2) \leqslant \log_3 3 \Leftrightarrow x+2 \leqslant 3 \Leftrightarrow x \leqslant 1$

C.S. $= ]-\infty, 1] \cap ]-2, +\infty[ = ]-2, 1]$

**10.2.**

$\log_2(\log x) > -1$

$D = \{x \in \mathbb{R}: \log x > 0 \wedge x > 0\} = ]1, +\infty[ \cap \mathbb{R}^+ = ]1, +\infty[$

$\log x > 0 \Leftrightarrow \log x > \log 1 \Leftrightarrow x > 1$

$\log_2(\log x) > -1 \Leftrightarrow \log_2(\log x) > \log_2 2^{-1} \Leftrightarrow \log x > \frac{1}{2} \Leftrightarrow$

$\Leftrightarrow \log x > \log 10^{\frac{1}{2}} \Leftrightarrow x > \sqrt{10}$

C.S. $= ]\sqrt{10}, +\infty[ \cap ]1, +\infty[ = ]\sqrt{10}, +\infty[$

**10.3.**

$\log_3 x + \log_3(x+6) < 3$

$D = \{x \in \mathbb{R}: x > 0 \wedge x+6 > 0\} = \mathbb{R}^+ \cap ]-6, +\infty[ = \mathbb{R}^+$

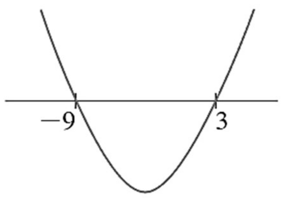
$\log_3 x + \log_3(x+6) < 3 \Leftrightarrow \log_3(x(x+6)) < \log_3 3^3 \Leftrightarrow x(x+6) < 27 \Leftrightarrow$

$\Leftrightarrow x^2 + 6x - 27 < 0 \Leftrightarrow x \in ]-9, 3[$

C.A.: $x^2 + 6x - 27 = 0 \Leftrightarrow$

$$x = \frac{-6 \pm \sqrt{36 - 4 \times 1 \times (-27)}}{2} \Leftrightarrow$$

$\Leftrightarrow x = -9 \vee x = 3$




C.S. $= ]-9, 3[ \cap \mathbb{R}^+ = ]0, 3[$

**10.4.**

$(\ln x)^2 + \ln x > 0$

$D = \{x \in \mathbb{R}: x > 0\} = \mathbb{R}^+$

$\ln x(\ln x + 1) > 0 \Leftrightarrow \ln x < -1 \vee \ln x > 0 \Leftrightarrow \ln x < \ln e^{-1} \vee \ln x > \ln 1 \Leftrightarrow$

$\Leftrightarrow x < \frac{1}{e} \vee x > 1$

C.S. $= \left]-\infty, \frac{1}{e}\right[ \cup ]1, +\infty[ \cap \mathbb{R}^+ = \left]0, \frac{1}{e}\right[ \cup ]1, +\infty[$

**10.5.**

$\log_{\frac{1}{5}}(x^2) \geqslant \log_{\frac{1}{5}} 4$

$D = \{x \in \mathbb{R}: x^2 > 0\} = \mathbb{R} \setminus \{0\}$

$\log_{\frac{1}{5}} x^2 \geqslant \log_{\frac{1}{5}} 4 \Leftrightarrow x^2 \leqslant 4 \Leftrightarrow x^2 - 4 \leqslant 0 \Leftrightarrow -2 \leqslant x \leqslant 2$

C.S. $= [-2, 2] \cap \mathbb{R} \setminus \{0\} = [-2, 0[ \cup ]0, 2]$

**10.6.**

$\log_2 (x+1)^2 < 0$

$D = \{x \in \mathbb{R}: (x+1)^2 > 0\} = \mathbb{R}\backslash\{-1\}$

$\log_2 (x+1)^2 < \log_2 1 \Leftrightarrow (x+1)^2 < 1 \Leftrightarrow x^2 + 2x + 1 - 1 < 0 \Leftrightarrow$

$\Leftrightarrow x^2 + 2x < 0 \Leftrightarrow -2 < x < 0$

C.S. $= ]-2, 0[ \cap \mathbb{R}\backslash\{-1\} = ]-2, -1[ \cup ]-1, 0[$

**11.**

$D = \{x \in \mathbb{R}: x > 0 \wedge 1 + \log x \neq 0\} = \mathbb{R}^+\backslash\left\{\frac{1}{10}\right\}$

$1 + \log x = 0 \Leftrightarrow \log x = -1 \Leftrightarrow x = \frac{1}{10}$

$\frac{\log x}{1+\log x} \leqslant 2 \Leftrightarrow \frac{\log x}{1+\log x} - 2\left(\frac{1+\log x}{1+\log x}\right) \leqslant 0 \Leftrightarrow$

$\Leftrightarrow \frac{\log x - 2 - 2\log x}{1+\log x} \leqslant 0 \Leftrightarrow -\frac{2+\log x}{1+\log x} \leqslant 0 \Leftrightarrow \frac{2+\log x}{1+\log x} \geqslant 0$

$2 + \log x = 0 \Leftrightarrow \log x = -2 \Leftrightarrow x = 10^{-2} \Leftrightarrow x = \frac{1}{100}$

$1 + \log x = 0 \Leftrightarrow \log x = -1 \Leftrightarrow x = 10^{-1} \Leftrightarrow x = \frac{1}{10}$

| | $-\infty$ | $\frac{1}{100}$ | | $\frac{1}{10}$ | $+\infty$ |
|---|---|---|---|---|---|
| $2 + \log x$ | $-$ | 0 | $+$ | $+$ | $+$ |
| $1 + \log x$ | $-$ | $-$ | $-$ | 0 | $+$ |
| $\frac{2+\log x}{1+\log x}$ | $+$ | 0 | $-$ | n.d. | $+$ |

$\frac{2+\log x}{1+\log x} \geqslant 0 \Leftrightarrow x \in \left]-\infty, \frac{1}{100}\right] \cup \left]\frac{1}{10}, +\infty\right[$

C.S. $= \left(\left]-\infty, \frac{1}{100}\right] \cup \left]\frac{1}{10}, +\infty\right[\right) \cap \mathbb{R}^+\backslash\left\{\frac{1}{10}\right\} = \left]0, \frac{1}{100}\right] \cup \left]\frac{1}{10}, +\infty\right[$