Nome do aluno | Nº | Data / / 20

Estudo da função logaritmo de base a

1. Tendo por base o gráfico de $f: \mathbb{R}^+ \to \mathbb{R}$, definida por $f(x) = \log_2 x$, represente graficamente:

   1.1. $g(x) = f(x) - 2$

   1.2. $g(x) = 1 - f(x)$

   1.3. $g(x) = f(-x)$

   1.4. $g(x) = f(x+1)$

   1.5. $g(x) = f^{-1}(x)$

2. Sejam $a \in \mathbb{R}$, $b \in \mathbb{R}^+\backslash\{1\}$ e $f(x) = a + b^{x+1}$.

   Sabendo que $f(0) = 2$ e $f(1) = 8$, determine $f^{-1}(0)$.

3. Determine o domínio e os zeros, se existirem, das funções definidas por:

   3.1. $f(x) = \log(3 - x)$

   3.2. $f(x) = \ln(3^x - 3)$

   3.3. $f(x) = \log_{\frac{1}{2}}|x|$

   3.4. $f(x) = \frac{\log_3(1-x^2)}{x}$

   3.5. $f(x) = \log_3\left(\frac{1-x^2}{x}\right)$

4. Considere as funções f e g definidas por:

   $$f(x) = \ln x \qquad g(x) = x^2 - 1$$

   Caracterize:

   4.1. $f \circ g$

   4.2. $g \circ f$

5. Sabendo que cada uma das funções seguintes é bijetiva, caracterize a sua função inversa:

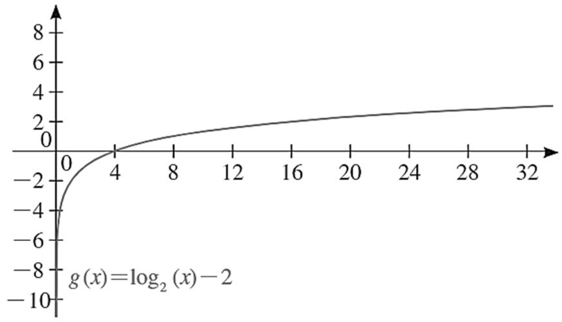
   5.1. $f(x) = \frac{e^{2x} - 1}{2}$

   5.2. $g(x) = 2\log(1 - x)$

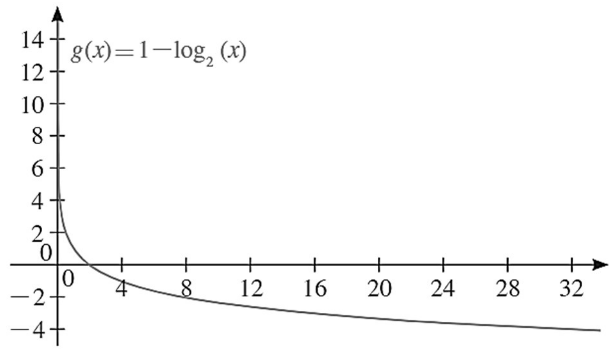
Soluções

1.

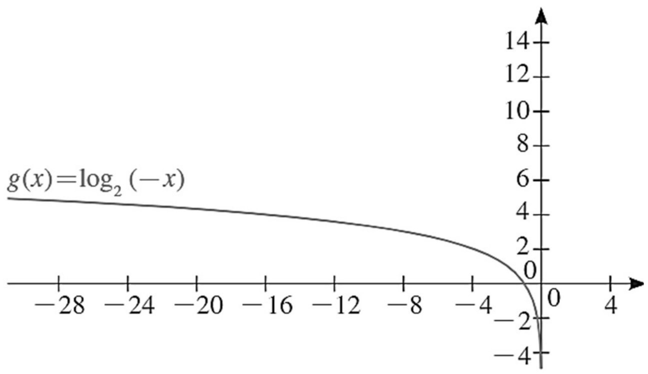
1.1.



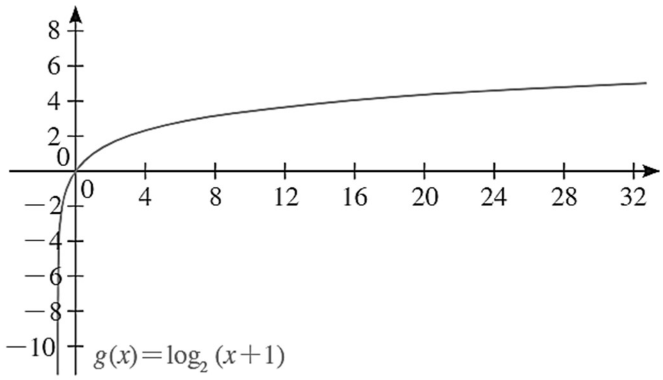
1.2.



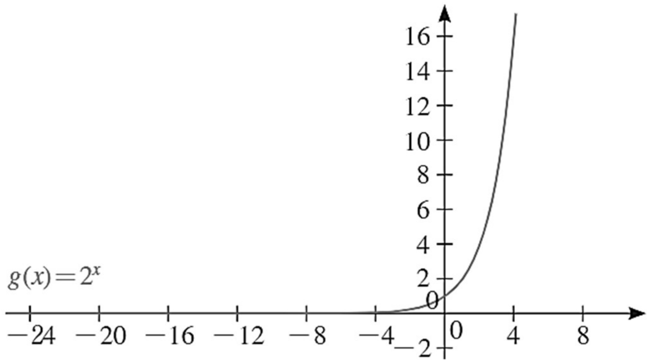
1.3.



1.4.



1.5.



2.

$f(0)=2 \Leftrightarrow a+b=2$ $\qquad$ $f(1)=8 \Leftrightarrow a+b^2=8$

$$\begin{cases} a+b=2 \\ a+b^2=8 \end{cases} \Leftrightarrow \begin{cases} a=2-b \\ 2-b+b^2=8 \end{cases} \Leftrightarrow \begin{cases} \_\_\_\_\_\_\_\_ \\ b^2-b-6=0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \_\_\_\_\_\_\_\_ \\ b=\dfrac{1 \pm \sqrt{1-4\times 1\times(-6)}}{2} \end{cases} \Leftrightarrow \begin{cases} a=-1 \\ b=-2 \vee b=3 \end{cases},$$

$b \in \mathbb{R}^+ \backslash \{1\}$, logo, $b=3$.

$f(x)=-1+3^{x+1}$

$f(x)=0 \Leftrightarrow -1+3^{x+1}=0 \Leftrightarrow 3^{x+1}=1 \Leftrightarrow x+1=0 \Leftrightarrow x=-1$

$f^{-1}(0)=1$

3.

3.1.

$f(x)=\log(3-x)$

$D_f=\{x \in \mathbb{R}: 3-x>0\}=\,]-\infty, 3[$

$f(x)=0 \Leftrightarrow \log(3-x)=0 \Leftrightarrow 3-x=1 \Leftrightarrow x=2$

Zeros: $\{2\}$

3.2.

$f(x)=\ln(3^x-3)$

$D_f=\{x \in \mathbb{R}: 3^x-3>0\}=\,]1,+\infty[$

$f(x)=0 \Leftrightarrow \ln(3^x-3)=0 \Leftrightarrow 3^x-3=1 \Leftrightarrow 3^x=4 \Leftrightarrow x=\log_3 4$

Zeros: $\{\log_3 4\}$

3.3.

$f(x)=\log_{\frac{1}{2}}|x|$

$D_f=\{x \in \mathbb{R}: |x|>0\}=\mathbb{R}\backslash\{0\}$

$f(x)=0 \Leftrightarrow \log_{\frac{1}{2}}|x|=0 \Leftrightarrow x=-1 \vee x=1$

Zeros: $\{-1, 1\}$

3.4.

$$f(x)=\frac{\log_3(1-x^2)}{x}$$

$D_f=\{x\in \mathbb{R}: 1-x^2>0 \wedge x\neq 0\}=\left]-1,1\right[\backslash\{0\}$

C.A.: $1-x^2>0 \Leftrightarrow x>-1 \wedge x<1$

$$f(x)=0\Leftrightarrow \frac{\log_3(1-x^2)}{x}=0\Leftrightarrow \log_3(1-x^2)=0 \wedge x\neq 0 \Leftrightarrow$$

$$\Leftrightarrow 1-x^2=1 \wedge x\neq 0 \Leftrightarrow x^2=0 \wedge x\neq 0\Leftrightarrow x=0 \wedge x\neq 0$$

Zeros: $\varnothing$

3.5.

$$f(x)=\log_3\left(\frac{1-x^2}{x}\right)$$

$$D_f=\left\{x\in\mathbb{R}: \frac{1-x^2}{x}>0 \wedge x\neq 0\right\}=\left]-\infty,-1\right[\cup\left]0,1\right[$$

	$-\infty$		-1		0		1	$+\infty$
$1-x^2$	$-$	$-$	0	$+$	$+$	$+$	0	$-$
x	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$
$\frac{1-x^2}{x}$	$+$	$+$	0	$-$	n.d.	$+$	0	$-$

$$f(x)=0\Leftrightarrow \log_3\left(\frac{1-x^2}{x}\right)=0\Leftrightarrow\frac{1-x^2}{x}=1\Leftrightarrow \frac{1-x^2-x}{x}=0\Leftrightarrow$$

$$\Leftrightarrow 1-x^2-x=0 \wedge x\neq 0\Leftrightarrow x=\frac{1\pm\sqrt{1-4\times(-1)\times 1}}{2\times(-1)} \wedge x\neq 0\Leftrightarrow$$

$$\Leftrightarrow x=\frac{1\pm\sqrt{5}}{-2} \wedge x\neq 0$$

$$\text{Zeros: }\left\{\frac{-1-\sqrt{5}}{2};\frac{-1+\sqrt{5}}{2}\right\}$$

4.

4.1.

$f\circ g(x)=f(g(x))=f(x^2-1)=\ln(x^2-1)$

$D_{f\circ g}=\{x\in\mathbb{R}: x\in D_g \wedge g(x)\in D_f\}=\left]-\infty,-1\right[\cup\left]1,+\infty\right[$

$g(x)\in D_f\Leftrightarrow x^2-1>0\Leftrightarrow x<-1 \vee x>1$

$f\circ g: \left]-\infty,-1\right[\cup\left]1,+\infty\right[\to\mathbb{R}$

$x\to\ln(x^2-1)$

4.2.

$g\circ f(x)=g(f(x))=g(\ln x)=\ln^2 x-1$

$D_{g\circ f}=\{x\in\mathbb{R}: x\in D_f\wedge f(x)\in D_g\}=\mathbb{R}^+$

$g\circ f: \mathbb{R}^+\to\mathbb{R}$

$x\to \ln^2 x-1$

5.

5.1.

$f(x) = \frac{e^{2x} - 1}{2}$ $\qquad\qquad D_f = \mathbb{R}$

$y = \frac{e^{2x} - 1}{2} \Leftrightarrow 2y + 1 = e^{2x} \Leftrightarrow 2x = \ln(2y + 1) \Leftrightarrow x = \frac{\ln(2y + 1)}{2}$

$f^{-1}(x) = \frac{\ln(2x + 1)}{2}$

$D_{f^{-1}} = \{x \in \mathbb{R}: 2x + 1 > 0\} = \left]-\frac{1}{2}, +\infty\right[$

$f^{-1}: \left]-\frac{1}{2}, +\infty\right[\to \mathbb{R}$

$x \to \frac{\ln(2x + 1)}{2}$

5.2.

$g(x) = 2\log(1 - x)$

$D_g = \{x \in \mathbb{R}: 1 - x > 0\} =]-\infty, 1[$

$y = 2\log(1 - x) \Leftrightarrow \frac{y}{2} = \log(1 - x) \Leftrightarrow 1 - x = 10^{\frac{y}{2}} \Leftrightarrow x = 1 - 10^{\frac{y}{2}}$

$g^{-1}(x) = 1 - 10^{\frac{x}{2}}$

$g^{-1}: \mathbb{R} \to]-\infty, 1[$

$x \to 1 - 10^{\frac{x}{2}}$